Straight line path following for formations of under-actuated marine surface vessels

Including ocean current compensation

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Abstract

In previous researches control techniques are defined to let underactuated marine surface vessels follow a straight line path in formation and to let a single vessel follow a path while disturbed by ocean current. Combining both techniques, this report investigates straight line path following of underactuated marine surface vessels in formation including compensation of the ocean current. In this sense decentralized controllers based on the Line Of Sight guidance law are defined for all participants. Furthermore the formation will move with a given velocity and structure and in the end of the report the results are validated by simulations.
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Chapter 1

Introduction

Autonomous control is a topic that gains a lot of attention in actual researches. It is of big interest in traffic and transportation, which mainly is based on cars, but also can be applied to vessels. A more specific topic is formation control of autonomous vehicles. Trucks are prepared for platooning, cars are equipped with ACC (Adaptive Cruise Control) and vessels can operate in formation by defining individual path following. The aim of this is to increase efficiency, safety and/or size of the task that has to be performed and especially for vessels it is practically useful, since formations of all desirable sizes and structures can be created. Vessels in formation can perform investigation or maintenance activities of the sea bed and by focusing on two vessels in formation, it enables activities such as repairing, maintenance or transporting freight during operation.

In [1] a method is described for underactuated marine vessels that have to operate in formation. Due to design constraints it is common that marine vessels are fully actuated at low speed, e.g. docking situations, while they are underactuated for common operation conditions, e.g. during maneuvering [2]. In [1], the vehicles are required to individually follow an assigned path while keeping given relative distances such that the vehicles move in a prescribed formation. The path following and relative distance regulation can be in principle dealt with separately, however due to the under actuation the actions will interfere. Control of the formation can be applied by either centralized or decentralized control. Here a centralized control strategy exists of regulation of all vessels via a central controller with full knowledge, while a decentralized control strategy exist of individual controllers for all vessels with only local knowledge.

The path following technique is widely investigated for single vessels with also methods that include the ocean current [3]. Here the controller compensates for the ocean current during the individual path following and since this is one of the two main control problems for path following in formation, it might be applicable in that case as well.

Combining a path following control strategy with ocean current compensation and a formation control strategy for straight line path motion based on along path relative distance regulation, this report investigates the formation control problem of underactuated marine vessels. For this decentralized controllers are defined for all participants, that, with some restrictions, are flexible in their way of communication. Furthermore the formation will move with a given velocity relative to the ocean current and a given structure of any form.
The report is organised as follows. Chapter 2 introduces a 3-DOF model which describes the vessel and defines the control objectives. In Chapter 3 a control technique is designed and some remarks are given on its application. Chapter 4 proofs stability in the single path following and Chapter 5 proofs stability in formation. This is validated by simulations in Chapter 6 and finally a conclusion and suggestions for future work are given in Chapter 7.
Chapter 2

Vessel model and Control objectives

2.1 Vessel model

To be able to describe the behaviour of a vessel that will be actuated in an imposed direction, a 3-DOF model is used as formulated in [4] to describe the behavior in the sense of orientation and velocity

\[
\begin{align*}
\dot{x} &= u \cos \psi - v \sin \psi = u_r \cos \psi - v_r \sin \psi + V_x \\
\dot{y} &= u \sin \psi + v \cos \psi = u_r \sin \psi + v_r \cos \psi + V_y \\
\dot{\psi} &= r
\end{align*}
\] (2.1)

with \( \eta = [x \ y \ \psi]^T \) the position and orientation in the global frame and \( \nu = [u \ v \ r] \) formulating the surge velocity, sway velocity and angular velocity of the vessel in a body fixed frame. Introducing the ocean current components in the global frame, \( V_x \) and \( V_y \), the model is formulated using the relative velocities \( \nu_r = [u_r \ v_r \ r]^T \). These velocities are relative with respect to the ocean current, which makes the description of movements in the body-fixed frame independent of the ocean current. Furthermore the definition of actual surge and sway velocities is enabled by this, since in practice the vessel moves relative to the ocean current and not relative to the global frame. To get the resulting behavior of the vessel in the global frame, the contribution of the ocean current simply is added to the relative velocities with respect to the direction as is done in Equation 2.1. By making use of the relative velocities, the behavior of the vessel is represented by a dynamic model, which enables modulations and simulations of the actual behavior

\[
M\ddot{\nu}_r + C(\nu_r)\nu_r + D\nu_r = Bf. \tag{2.2}
\]

Here \( M \) is the Mass and Inertia matrix, \( C \) the Coriolis matrix, \( D \) the damping matrix, \( B \) the actuator configuration matrix and \( f = [T_u \ T_r]^T \) a vector containing the control inputs. From \( f \) it can be concluded that the system only exist of two control inputs \( T_u \) for surge control and \( T_r \) for yaw control. The matrix \( B \) maps them such a way that these physically meaningless control values are formulated to real control forces and moments defined as thruster force and rudder angle. A general form for the matrices used in the dynamical model is given by
\[ M = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix} = M^T > 0 \]

\[ C = \begin{bmatrix} 0 & 0 & -m_{22}v_r - m_{23}r \\ 0 & 0 & m_{11}u_r \\ m_{22}v_r + m_{23}r & -m_{11}u_r & 0 \end{bmatrix} \]

\[ D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix} \]

\[ B = \begin{bmatrix} b_{11} \\ 0 \\ 0 \end{bmatrix} \]

with structures that normally hold for marine surface vessels [2]. This is based on the assumption that most of them are port-starboard symmetric, which explains the structure of matrix \( M \) and \( D \) and by having those, \( C \) is formulated in its actual structure. To get the model in this form a few more assumptions have been made, also regarding the dynamics

- The origin of the body fixed frame is located at a distance \( x_g^* \) from the vessels’ center of gravity on the center-line of the ship. This is achieved by moving the origin of the body fixed frame along its x-axis until the rudder gives only a rotational moment and no sway force [2] [5].
- The hydrodynamic damping is linear [3].
- The ocean current in the global frame is constant, irrotational and bounded [3].
- \( Y(u_r) \), which is a measure of sensitivity of the sway velocity on itself, is defined to be strictly negative. This makes sense, since \( Y(u_r) > 0 \) defines a vessel that is unstable in sway and this is an unfeasible condition in nature.

By reformulating the model of 2.2 as a function for \( \dot{v}_r \), the dynamics of the model are defined by

\[ \dot{u}_r = F_{u_r}(v_r, r) - \frac{d_{11}}{m_{11}} u_r + \tau_u \]

\[ \dot{v}_r = X(u_r)r + Y(u_r)v_r \]

\[ \dot{r} = F_r(u_r, v_r, r) + \tau_r \]

with new control inputs \( \tau_u \) and \( \tau_r \) that are directly extracted from the model by applying \( \begin{bmatrix} \tau_u & \tau_r \end{bmatrix}^T = M^{-1}Bf \) [3]. The terms \( F_{u_r}(v_r, r), X(u_r), Y(u_r) \) and \( F_r(u_r, v_r, r) \) are just collections of the nonlinear dynamics, which for sake of simplicity are not fully defined in the equations. For completeness the specific expressions are given in Appendix A.

### 2.2 Control objectives

Two main control approaches exist to regulate the behavior of a system consisting of multiple participants. Either one central controller demands all operations that should be carried out
by the single operators or every single operator has to take care of its own control decisions. The first method requires full knowledge about the behavior of all participants and for this reason may be more complex and less flexible. For this reason, in this report decentralized control laws are defined to guarantee that every vessel participates correctly in the formation. To achieve this every vessel has to follow its own path and has to adapt its relative position to that of all others in the along path direction. The control objectives are defined by

\[
\lim_{t \to \infty} y_j(t) - D_j = 0 \tag{2.4}
\]

\[
\lim_{t \to \infty} \psi_j(t) - \psi_{d_j}(t) = 0 \tag{2.5}
\]

\[
\lim_{t \to \infty} x_j(t) - x_i(t) - d_{ji} = 0 \tag{2.6}
\]

\[
\lim_{t \to \infty} u_{r_j}(t) - u_{r_d}(t) = 0 \tag{2.7}
\]

with \(D_j\) the \(y\)-position in the global frame of the linear track that has to be tracked by vessel \(j\) and is aligned with the \(y\)-axis of the global frame. \(\psi_{d_j}\), \(u_{r_d}(t)\) and \(d_{ji}\) are the desired angle of vessel \(j\), the desired surge velocity of the formation and the desired inter vessel distance between vessel \(j\) and \(i\). Furthermore \(u_{r_j}\) satisfies

\[
u_{r_j}(t) \in [U_{\text{min}}, U_{\text{max}}] \tag{2.8}
\]

to guarantee that the vessel moves at least with a velocity at which it is controllable. The vessel is uncontrollable at a surge velocity \(u_{r_j}(t) = 0\) since steering will not affect its behavior at all. However in this case also the ocean current is taken into account and \(U_{\text{min}}\) is assigned such that it will overcome this, \(U_{\text{min}} > \|V\|\), meaning that the vessel will move forward in the global frame. Next to this \(U_{\text{max}}\) guarantees that the vessel does not have to go faster than its maximum speed, which is applicable since \(u_{r}\) defines the real surge velocity. The first two control objectives 2.4 and 2.5 can be achieved by every vessel individually, while 2.6 and 2.7 require information about the along-path position of the other vessels. For achieving them the complete formation has to be taken into account and single vessels will have to communicate about their position. For control objective 2.7 this is not represented explicitly, however in defining the desired surge velocity of the formation during its formation, the relative distances of all vessels have to be taken into account.

### 2.3 Communication

As an extension for the model of the single vessel, the model of multiple vessels has to take into account communication. This is the most substantial difference since all controllers are performing their tasks decentralized. It means that every vessel manages its own path following independently and for this task knowledge of the parameters of the vessel itself are sufficient. However to control the formation, parameters of other vessels are required as well, to be able of maintaining the desired inter-vessel distances. The communication network required for this can be represented by digraph \(G = (\mathcal{V}, \mathcal{E})\) with a set of vertices \(\mathcal{V}\) and a set of edges \(\mathcal{E}\). Here the vertices in set \(\mathcal{V}\) represent the \(n\) vessels and the edges in set \(\mathcal{E}\) represent the available communication links. To obtain a proper communication network it is formulated that the formation has at least one \textit{Globally reachable vertex}, meaning that information of one vessel can reach every other vessel, not taking into account the way it has to travel. This means that no direct communication between every vessel is required and that one directional communication also can be useful. By stating this a wide variety of vessels can be used in the formation, just having the ability of sending and/or receiving information. Figure A.1 represents a digraph with a globally reachable vertex.
Chapter 3

Controller design and application

3.1 LOS angle, compensating lateral ocean current

In order to make each single vessel follow its own path, a decentralized control law is defined. First of all a technique is defined to let the vessel navigate towards the line it has to follow, called track in this report. This is achieved by making use of the LOS (Line Of Sight) guidance law, which is based on defining an orientation angle such that the vessel moves towards the track by focusing on a point ahead of the vehicle and placed on the track. This situation is represented by Figure 3.1(a) in which $\psi_d$ defines the orientation required to move towards the point defined by the look ahead distance $\Delta$. The route traveled by the vessel is called its path.

By taking the ocean current into account, the vessel will have to adapt its orientation to compensate for this as is represented in Figure 3.1(b) by the term $\Delta \mu$. This makes the vessel orientate towards the direction the lateral ocean current $V_y$ is comming from. The factor $\mu$ is an extra control parameter which is introduced to compensate the lateral ocean current component $V_y$ at all times, even when the vessel has already converged to the track as is represented in Figure 3.1(c). Implementation is simply achieved by adding $\mu$ to the approach.
that is used in [1] to determine the desired orientation \( \psi_{d_j} \) of vessel \( j \), the equation for \( \psi_{d_j} \) is given by

\[
\psi_{d_j} = -\tan^{-1}\left(\frac{e_j}{\Delta_j} + \mu_j\right), \quad e_j = y_j - D_j.
\] (3.1)

Here \( \Delta_j \) is the look ahead distance, which is a control parameter. \( e_j \) is the cross-track error and \( \mu_j \) is a factor, still to be designed, that compensates for the lateral ocean current \( V_y \). All parameters are defined for vessel \( j \) and the definition of \( \mu_j \) is based on the derivative of the cross-track error

\[
\dot{e}_j = \dot{y}_j = u_{r_j} \sin \psi_j + v_{r_j} \cos \psi_j + V_y
\] (3.2)

and simply compensates the term \( V_y \) such that the cross track error becomes independent of the (lateral) ocean current. Introducing the vector \( \xi_j = [\bar{u}_{r_j}, \dot{\psi}_j, \dot{\bar{r}}_j] \) with tracking errors \( \bar{u}_{r_j} = u_{r_j} - u_{r_dj} \), \( \dot{\psi}_j = \dot{\psi}_j - \dot{\psi}_{d_j} \) and \( \dot{\bar{r}}_j = r_j - \dot{\bar{r}}_{d_j} \) and defining \( v_{r_dj} = v_{r_dj} \), since \( v_{r_dj} = 0 \), Equation 3.2 can be reformulated to

\[
\dot{e}_j = (\bar{u}_{r_j} + u_{r_dj}) \sin(\psi_j + \psi_{d_j}) + v_{r_j} \cos(\psi_j + \psi_{d_j}) + V_y
\]

\[
= u_{r_dj} \sin(\psi_{d_j}) + v_{r_j} \cos(\psi_{d_j}) + V_y + h_c^T \xi_j
\]

\[
= u_{r_dj} \sin \left( -\tan^{-1}\left(\frac{e_j + \Delta_j \mu_j}{\Delta_j}\right) \right) + v_{r_j} \cos \left( -\tan^{-1}\left(\frac{e_j + \Delta_j \mu_j}{\Delta_j}\right) \right) + V_y + h_c^T \xi_j
\]

\[
= -\frac{e_j u_{r_dj}}{\sqrt{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2}} + \frac{\Delta_j v_{r_j}}{\sqrt{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2}} - \frac{\Delta_j \mu_j u_{r_dj}}{\sqrt{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2}} + V_y + h_c^T \xi_j.
\] (3.3)

Here the term \( h_c \) is defined as

\[
h_c = \begin{bmatrix}
\sin \tilde{\psi} & -\frac{e_j \mu_j}{\sqrt{(\mu \Delta + e_j)^2 + \Delta^2}} \cos \tilde{\psi} + \frac{\Delta_j \mu_j}{\sqrt{(\mu \Delta + e_j)^2 + \Delta^2}} \sin \tilde{\psi} \\
\frac{u_{r_dj} \Delta}{\sqrt{(\mu \Delta + e_j)^2 + \Delta^2}} + \frac{v_{r_j} (e_j + \mu \Delta)}{\sqrt{(\mu \Delta + e_j)^2 + \Delta^2}} & \cos \tilde{\psi} - 1
\end{bmatrix}
\]

(3.4)

and from this it can be seen that when the vessel is tracking the desired values \( u_{r_d}, \psi_d \) and \( r_d \), which is equivalent to \( h_c^T \xi_j = 0 \), the lateral ocean current is compensated by \( \mu_j \), that can be designed as the solution of the following equation

\[
-\frac{\Delta_j \mu_j u_{r_dj}}{\sqrt{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2}} + V_y = 0.
\] (3.5)

This results in the cross track error dynamics \( \dot{e}_j \) being a function of \( e_j \) and \( v_{r_d} \) only, which are the variables of the nominal system of a cascaded system as will be explained in Chapter 4. Additionally the next section introduces \( u_{r_{d_j}} \), which is a new desired surge velocity that takes into account the positioning relative to other vessels and for this reason will replace \( u_{r_dj} \) when taking into account multiple vessels. As a result of this \( \mu_j \) eventually is defined as

\[
\mu_j = \frac{V_y \left( e_j V_y + \sqrt{(e_j^2 + \Delta_j^2) u_{r_{d_j}}^2 - \Delta_j^2 V_y^2}\right)}{\Delta_j \left( u_{r_{d_j}} - V_y^2\right)}.
\] (3.6)
3.2 Observer

Since the control of the LOS angle is dependent on $V_y$ via the term $\mu_j$, it is required to know the actual value of the ocean current such that the desired angle is determined correctly. This problem is solved by obtaining estimations of the velocities of the ocean current in lateral direction, $\hat{V}_y$, and longitudinal direction, $\hat{V}_x$, with the Luenberger observer

\[
\dot{\hat{x}}_j = u_{rj} \cos(\psi_j) - v_{rj} \sin(\psi_j) + \hat{V}_x + K_{11}(x_j - \hat{x}_j),
\]

\[
\dot{\hat{y}}_j = u_{rj} \sin(\psi_j) + v_{rj} \cos(\psi_j) + \hat{V}_y + K_{12}(y_j - \hat{y}_j),
\]

\[
\dot{\hat{V}}_x = K_{21}(x_j - \hat{x}_j),
\]

\[
\dot{\hat{V}}_y = K_{22}(y_j - \hat{y}_j).
\]

Here $\hat{x}_j$ and $\hat{y}_j$ are estimations of the position of the vessel $j$ and $K_{11}$, $K_{12}$, $K_{21}$ and $K_{22}$ are positive constants. Implementing an observer makes it possible to determine these values without the need to measuring the ocean current, which might be complex and expensive.

3.3 Commanded surge velocity

The vessel only can be actuated in two degrees of freedom and since the compensation of the lateral ocean current and the control of the path following both is achieved by the yaw angle, it makes sense to control the along path positioning with the surge velocity. To be able to cope with formations a commanded surge velocity is defined as in [1]

\[
u_{r_{cij}} = u_{rd}(t) - g \left( \sum_{i \in A_j} (x_j - x_i - d_{ji}) \right)
\]

with $g$ a continuously differentiable function with saturation values $\pm a$. The addition of this term enables the vessels to accelerate/decelerate up to a value $a$ to arrange the alignment of the formation. A typical choice for a saturation-like function is $g(x) = 2a/\pi \tan^{-1} x$, which also is used in the simulations worked out in Chapter 6. $g(x)$ is designed to take account of the along path position of vessel $j$ and the positions of all other vessels $i \in A_j$ and compares their actual inter-vessel distances with the desired ones. The set $A_j$ consists of all vessels $i$ communicating information to vessel $j$. The value of $g$ will converge to zero when the formation is formed, resulting in the same desired surge velocity for all vessels. By implementing the commanded surge velocity, the dynamics in along path direction will look like

\[
\dot{\hat{x}}_j = u_{rj} \cos(\psi_j) - v_{rj} \sin(\psi_j) + V_x,
\]

\[
(\hat{u}_{r_{cij}} + \hat{u}_{rd}) \cos(\psi_{d_j} + \tilde{\psi}_j) - v_{rj} \sin(\psi_{d_j} + \tilde{\psi}_j) + V_x
\]

\[
(\hat{u}_{r_{cij}}) \cos(\psi_{d_j}) - v_{rj} \sin(\psi_{d_j}) + V_x + h_x \xi_j + h_x^T \xi_j
\]

with $h_x$ a term multiplied by $\xi_j$, which contains the tracking errors. These go to zero for $t \to \infty$, which eventually results in $h_x^2 \xi_j \to 0$ for $t \to \infty$. The exact formulation is given in Appendix B.
3.4 Controllers

To control the path following of the vessels, both control inputs mentioned in 2.3a and 2.3c have to be defined. For this the yaw controller \( \tau_{rj} \) and surge controller \( \tau_u \) are given by

\[
\tau_{rj} = -F_r + \dot{\psi}_{dy} - k_{\psi j} (\psi_j - \psi_{dy}) - k_{rj} (r_j - \dot{\psi}_{dy}) \\
\tau_u = -F_u + \dot{u}_{rcj} - k_{u} (u_{rj} - u_{rcj})
\] (3.10)

with \( \psi_{dy} \) and \( u_{rcj} \) the desired values in yaw and surge velocity as defined before and \( \dot{\psi}_{dy} \) the derivative of \( \psi_{dy} \), of which the derivation will be explained in the next section. The control gains \( k_{\psi j} \) and \( k_{rj} \) are defined to be strictly positive, making \( \tau_{rj} \) become a feedback linearizing PD-controller that makes the yaw angle \( \psi_j \) exponentially converge to the desired LOS angle \( \psi_{dy} \). Similarly \( \tau_u \) becomes a feedback linearizing P-controller that makes the surge velocity \( u_{rj} \) exponentially converge to the commanded surge velocity \( u_{rcj} \). To achieve this the control gain \( k_u \) is defined to be strictly positive.

3.5 Observer application

Applying the observer in the simulations gives perfect results, however proving stability in formation, as is done in Chapter 5, is complicated a lot since \( \psi_{dy} \) is not equal to \( r_{dy} \) as long as the observer is not converged. This is based on the fact that the formulation of the desired angular velocity, \( r_{dy} \), is a function of the ocean current \( V_x \) and \( V_y \). These exact values however are not available for the controller since no measurements are executed, for which the reasons are explained in the previous sections. To be able to compensate for the ocean current the estimated values \( \hat{V}_{xj} \) and \( \hat{V}_{yj} \) are created by the observer. These values converge exponentially fast to the actual values, however during convergence there will be a difference in the formulations of \( \psi_{dy} \) and \( r_{dy} \) which are stated by

\[
r_{dy} = \frac{\Delta_j^2}{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2} \left( \frac{1}{\Delta_j} + \frac{\partial \mu_j}{\partial \psi_{dy}} \right) \dot{\psi}_{dy} + \frac{\partial \mu_j}{\partial u_{rcj}} \dot{u}_{rcj} + \frac{\partial \mu_j}{\partial v_{rcj}} \dot{v}_{rcj} \] (3.11)

\[
\dot{\psi}_{dy} = \frac{\Delta_j^2}{\Delta_j^2 + (e_j + \Delta_j \mu_j)^2} \left( \frac{1}{\Delta_j} + \frac{\partial \mu_j}{\partial \psi_{dy}} \right) \dot{\psi}_{obs} + \frac{\partial \mu_j}{\partial u_{rcj}} \dot{u}_{rcj} + \frac{\partial \mu_j}{\partial V_{yj}} \dot{V}_{yj} \] (3.12)

with \( \dot{\psi}_{obs} \) as defined in Equation 3.2 and

\[
\dot{\psi}_{obs} = u_{rj} \sin \psi_j + v_{rj} \cos \psi_j + \dot{V}_{yj},
\] (3.13)

This means that \( \mu_j \) applied to Equation 3.13 becomes a function of \( \hat{V}_{yj} \) regarding Equation 3.5. As a result of the different definitions, the tracking error dynamics cannot be defined GES since it no longer holds that \( \dot{\psi}_j = \hat{r}_j \). Instead of this \( \dot{\psi}_j = \hat{r}_j + f(\hat{V}_y) \) with \( f(\hat{V}_y) \) a vanishing term in the difference in the actual and observer value of \( V_y \). The difficulty is proving that the term with \( \hat{V}_{yj} \) indeed is vanishing regardless to all other parameters and dynamics. To avoid this part of the proof an assumption is made that the values of \( \dot{\psi}_{dy} \) and \( \dot{\psi}_{dj} \) are derived perfectly by filtering the signal \( \psi_{dy} \) using a low-pass filtering reference model [4]. As a result of this \( \dot{\psi}_{dy} \) is simply defined as \( r_{dy} \), which is worked out as

\[
\dot{\psi}_{dy} = e_j \frac{\Gamma_{j} u_{rcj} \Delta_j}{\sqrt{(\mu_j \Delta_j + e_j)^2 + \Delta_j^2}} - v_{rj} \frac{\Gamma_{j} \Delta_j^2}{\sqrt{(\mu_j \Delta_j + e_j)^2 + \Delta_j^2}} + \frac{V_y \Delta_j}{\sqrt{(\Delta_j^2 + e_j^2) u_{rcj}^2 - \Delta_j^2 V_y^2}} \dot{u}_{rcj}
\] (3.14)
with \( \Gamma_j \) a multiplication factor that takes into account the ocean current

\[
\Gamma_j = \frac{u_{rcj}^2}{u_{rcj}^2 - V_y^2} \left( \frac{e_j V_y}{\sqrt{(\Delta_j^2 + e_j^2)u_{rcj}^2 + \Delta_j^2 V_y^2}} + 1 \right). \tag{3.15}
\]

This term is defined by \( \Gamma_j = \left( 1 + \frac{\Delta \partial \mu}{\partial e_j} \right) \) as can be reformulated from Equation 3.11. For the model that does not take into account the ocean current this term simply is 1, since \( \mu \) does not exist and for this reason \( \Gamma \) is the link between the model that includes the ocean current and the one that does not. This will be explained some more explicit in Chapter 4.

### 3.6 Physical restrictions

As mentioned in Equation 2.8, \( u_{rcj} \) is limited with a maximum value and next to this the minimum relative surge velocity has to be bigger than zero to let the vessel be controllable. To reformulate this condition to a condition for the commanded relative surge velocity \( u_{rcj} \), Equation 3.8 is reformulated to its limits

\[
u_{r,\text{max},\text{min}} = u_{rd}(t) \pm a. \tag{3.16} \]

This results in a condition for \( u_{rd} \) as a function of the minimum and maximum velocities of the vessel and the saturation value \( a \)

\[
u_{rd}(t) \in \left[ U_{\text{min}} + a, U_{\text{max}} - a \right]. \tag{3.17} \]

Note: For convenience the minimum surge velocity has to be bigger than \( V_y \) to overcome the lateral ocean current, making it possible to move in the global frame towards the track. For this reason the assumption is made

\[
U_{\text{min}} > \| V_y \|. \tag{3.18}
\]

Next to the velocity two more terms have to be taken into account when defining stability of the vessel. The terms \( X(u_{rcj}) \) and \( Y(u_{rcj}) \) as mentioned in Equation 2.3b are measurements for the sensitivity of the sway velocity on the yaw rate and sway velocity itself respectively. Since both are a function of \( u_{rcj} \), extreme values are used to determine the sensitivity of the vessel on both disturbances in the extreme case

\[
X_{j,\text{max}} = \max_{u_{rcj} \in [U_{\text{min}}, U_{\text{max}}]} \left| X_j(u_{rcj}) \right| \]
\[
Y_{j,\text{min}} = \min_{u_{rcj} \in [U_{\text{min}}, U_{\text{max}}]} \left| Y_j(u_{rcj}) \right| \tag{3.19}
\]

The relation \( \frac{X_j}{Y_j} \) is a measure for the sensitivity and thus also for the stability. The relation that follows from this is that the stability simply improves by enlarging \( \Delta_j \), which for a minimum \( \Delta_j \) results in a stability condition. In the sequel of this report the condition for \( \Delta_j \) will be explained in some more detail for two cases applied on a single vessel. One is based on a constant desired surge velocity and the other on a variable one.
Chapter 4

Stability proof for path following for a single vessel

4.1 Constant desired surge velocity \( u_{rd} \)

During the convergence to the track, the vessel can be defined as a cascaded system existing of a nominal system describing the sway dynamics \( \begin{bmatrix} \dot{e} \\ \dot{v}_r \end{bmatrix} \) and a perturbing system describing the dynamics of the tracking errors \( \dot{\xi} \). For ease the subscript \( j \), denoting the specific vessel, is left out of this proof since a single vessel is taken into account. This results in

\[
\begin{bmatrix} \dot{e} \\ \dot{v}_r \end{bmatrix} = A(e) \begin{bmatrix} e \\ v_r \end{bmatrix} + H(e, v_r, \xi) \xi 
\]

(4.1)

\[
\dot{\xi} = \Lambda \xi 
\]

(4.2)

with

\[
A(e) = \begin{bmatrix} -u_{rd} \sqrt{(\mu e_+ e)^2 + \Delta^2} & \Delta \sqrt{(\mu e_+ e)^2 + \Delta^2} \\ X(u_{rd}) \Gamma \Delta \sqrt{(\mu e_+ e)^2 + \Delta^2} & Y(u_{rd}) - \frac{X(u_{rd}) \Gamma \Delta^2}{\sqrt{(\mu e_+ e)^2 + \Delta^2}} \end{bmatrix} 
\]

(4.3)

and \( \Gamma \) a multiplication factor that takes into account the ocean current as defined in Equation 3.15. The introduction of this term is the only difference with respect to the model that does not take into account the ocean current [1]. In Equation 3.15 it can be seen that \( \Gamma \) is equal to one for \( V_y = 0 \), which confirms that this model just is an extension to the model in which no ocean current is taken into account. Furthermore \( \Gamma \) is always positive, taking assumption 3.18 into account, which means that it will not make the system become unstable by changing the sign of the terms. Next to this, in this proof \( \mu \) is a function of \( u_{rd} \) in stead of \( u_{rc} \) to meet the condition of constant desired surge velocity which is naturally for a single vessel. The second term in Equation 4.1, \( H \), is the interconnection term between both dynamics

\[
H(e, v_r, \xi) = \begin{bmatrix} 0 & 1 \\ -X(u_{r}) \frac{\Gamma \Delta}{(\mu e_+ e)^2 + \Delta^2} & 0 \end{bmatrix} \begin{bmatrix} h^T_e \\ h^T_{vr} \end{bmatrix}
\]

(4.4)

with \( h_e \) and \( h_{vr} \) existing of terms that are formulated such that they will be multiplied by the tracking errors \( \xi \), as can be seen in Equation 4.1. This means that the complete term of \( H \) vanishes for \( \xi \to 0 \), which will be the case when the vessel is converged to the desired velocity and orientation. The definition for \( h_e \) is given in Equation 3.4 and \( h_{vr} \) is given by
\[
\begin{align*}
\mathbf{h}_{\text{re}} &= \begin{bmatrix}
X(u_{rd} + \tilde{u}_r) - X(u_{rd}) \left( \frac{\Gamma u_{rd} e}{\sqrt{(\mu \Delta + \epsilon)^2 + \Delta^2}} - \frac{\Gamma \Delta^2 v_r}{\sqrt{(\mu \Delta + \epsilon)^2 + \Delta^2}} \right) & + v_r \frac{Y(u_{rd} + \tilde{u}_r) - Y(u_{rd})}{u_r^n} \\
0 & X(u_{rd} + \tilde{u}_r)
\end{bmatrix}.
\end{align*}
\] (4.5)

Taking into account a constant desired surge velocity, the structure of the nominal system will be the same as the structure used in [1], where no ocean current was taken into account. As a result of this the proof can be worked out in a similar way, as will be done in this section.

To prove stability for the cascaded system, Theorem 2 of [6] is used, which is based on three assumptions that for this specific case can be formulated as below. In this formulation \(x_1 = [e \ v_r]^T, f_1(t, x_1) = A(x_1), x_2 = \xi \) and \( f_2(t, x_2) = \Lambda x_2 \) as defined in Equation 4.10.

1. When \( \dot{x}_1 = f_1(t, x_1) \) is GUAS, which can be proven by a Lyapunov function of the form
\[
V(t, x) = \sum_{i=1}^{n} k_i |x_i|^{p_i}
\]
with \( p_i \in (1, \infty), k_i > 0, i \in \mathbb{N} \), the nominal system is globally uniformly asymptotically stable.

2. The interconnection term has to be bounded, which mathematically is formulated by
\[
||H(e, v_r, u_{rd}, \xi)|| \leq \theta_1(|\xi|)(|e| + |v_r|) + \theta_2(|\xi|)
\]
where \( \theta_1, \theta_2 \) are continuous non-negative functions. It actually means that \( H(e, v_r, u_{rd}, \xi) \) can at most grow linearly with the perturbing terms \( e \) and \( v_r \).

3. To guarantee stability for the perturbing system, \( \dot{x}_2 \) has to be globally uniformly asymptotically stable and
\[
\int_{t_0}^{\infty} ||x_2(t, t_0, x_2(t_0))|| dt \leq \phi(||x_2(t_0)||)
\]
holds for every \( t_0 \geq 0 \). Reformulating this to
\[
\int_{t_0}^{\infty} ||x_2(t, t_0, x_2(t_0))|| dt \leq \int_{t_0}^{\infty} ||x_2(t_0)|| e^\lambda dt \leq \phi(||x_2(t_0)||)
\]
shows that this is fulfilled by a globally exponentially stable perturbing system.

In order to show that Theorem 2 in [6] is satisfied by our system, we start proving that assumption 1 of Theorem 2 in [6] is fulfilled. To this purpose we define the following Lyapunov function candidate
\[
V = \frac{1}{2} e^2 + \frac{1}{2} \kappa v_r^2, \text{ with } \kappa > 0 \text{ and the derivative defined by}
\]
\[
\dot{V} \leq -\Delta \left( \frac{\sqrt{u_{rd}}|e|}{\sqrt{(\mu \Delta + \epsilon)^2 + \Delta^2}} - \frac{\alpha |v_r|}{\sqrt{u_{rd}}} \right)^2 - \Delta \frac{(\alpha - 1)\alpha u_{rd}}{u_{rd}} |v_r|^2
\] (4.6)
which clearly is smaller than zero by defining \( \alpha > 1 \). The steps of deriving this are worked out in Appendix C and are similar to the procedure applied in [1]. From the derivation it follows that the definition of \( \alpha \) is given by
\[
\alpha = \frac{Y_{\min} \Delta - X_{\max} \Gamma}{X_{\max} \Gamma}.
\]
from which, after combining it with the condition $\alpha > 1$, the lower bound can be extracted for the look ahead distance. This means that choosing the control parameter $\Delta$ bigger than this value will guarantee stability in yaw and sway dynamics

$$\Delta > 2 \frac{X_{\text{max}} \Gamma}{Y_{\text{min}}}.$$ (4.7)

Unfortunately $\Gamma$ is a function of $\Delta$ which not simply can be isolated from all terms. To deal with this $\Gamma$ is reformulated for an extreme case in which the cross-track error goes to infinity, leading to a saturated maximum value

$$\Gamma_{\text{max}} = \frac{u_{rd}(V_{y} + u_{rd})}{u_{rd}^{2} - V_{y}^{2}}$$ (4.8)

independent of $\Delta$, making it possible to define a proper lower bound for $\Delta$ such that assumption 1 is satisfied by

$$\Delta > 2 \frac{X_{\text{max}} \Gamma_{\text{max}}}{Y_{\text{min}}}.$$ (4.9)

The next step in the proof is assumption 2, from which easily can be confirmed that it holds by checking Equations 4.4, 3.4 and 4.5. To prove assumption 3, the perturbing system

$$\dot{\xi} = \begin{bmatrix} -(k_{u} + \frac{d_{11}}{m_{11}}) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -k_{\psi} & -k_{r} \end{bmatrix} \xi$$ (4.10)

has to be GES. Since the matrix is Hurwitz, this condition also is satisfied. This results in an achievement of all three assumptions, meaning that the cascaded system is GUAS.

### 4.2 Variable desired surge velocity $u_{rc}$

In the previous section it was assumed that a single vessel performs the path following procedure under a constant desired surge velocity, meaning that the aim is to move with the same velocity over the complete trajectory. However in this report we are interested in single vessels moving in formation and for this they need the ability to vary their surge velocity to correct for errors in the positioning relative to other vessels. In order to do this the desired surge velocity has to be variable which is achieved by replacing $u_{rd}$ by $u_{rc}$, defined in Equation 3.8. The variable surge velocity however affects the nominal system of Equation 4.1 some more than simply replacing this term, since a term of $\dot{u}_{rc}$ is introduced and for this reason was neglected

$$\begin{bmatrix} \dot{e} \\ \dot{v}_{r} \end{bmatrix} = A(e, u_{rc}) \begin{bmatrix} e \\ v_{r} \end{bmatrix} + H(e, v_{r}, u_{rc}, \xi) \xi + \begin{bmatrix} 0 \\ \frac{V_{y} \Delta}{\sqrt{\Delta^{2} + e^{2}}} \frac{u_{rc}}{V_{y}^{2}} \end{bmatrix}.$$ (4.11)

This extra term is created by the dependency of $\psi_{d}$ on $u_{rc}$ as can be seen by combining Equations 3.1 and 3.6 or by checking the derivative in Equation 3.11 in which $r_{d}$ is equal to $\dot{\psi}_{d}$. The derivative of $\psi_{d}$ now is a function of $u_{rc}$, which via Equation 2.3b is introduced to the sway dynamics. Stability for the system with as nominal system the sway dynamics of Equation 4.11 can be proven by making use of the assumptions of Theorem 2 of [6] as defined in the previous section. With respect to the previous section, assumption 1 needs to be evaluated again while assumption 2 and 3 automatically are satisfied since the perturbing
system and the interconnection stay unchanged. Furthermore \( f_1(t, x_1) = A(e, u_{rc})x_1 \) is defined the same as in Equation 4.3 was done for a constant surge velocity. For this reason the same Lyapunov function candidate can be used, with derivative 4.6. By taking into account that \( u_{rc} \) and \( \dot{u}_{rc} \) satisfy

\[
|V| < U_{\text{min}} \leq u_{rc} \leq U_{\text{max}} \quad \dot{u}_{rc} \leq \ddot{u}_{rc} \leq \overline{u}_{rc}
\]

the derivative of the Lyapunov function is defined slightly different and with one extra term containing \( \dot{u}_{rc} \)

\[
\dot{V} \leq -\Delta \left( \frac{\sqrt{U_{\text{min}}|e|}}{(\mu \Delta + e)^2 + \Delta^2} - \frac{\alpha|v_r|}{\sqrt{U_{\text{min}}}} \right)^2 - \Delta \frac{(\alpha - 1) \alpha}{U_{\text{min}}} |v_r|^2 + \kappa \frac{|V_g|u_{rc}}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r|
\]

with \( \alpha \) now defined as a function of \( U_{\text{min}} \) and \( U_{\text{max}} \)

\[
\alpha = \frac{U_{\text{min}}(Y_{\text{min}} \Delta - X_{\text{max}} \Gamma)}{U_{\text{max}} X_{\text{max}} \Gamma}
\]

and a new lower bound defined for the look ahead distance, considering \( \Gamma_{\text{max}} \)

\[
\Delta > \frac{X_{\text{max}} \Gamma_{\text{max}}}{Y_{\text{min}}} \left(1 + \frac{U_{\text{max}}}{U_{\text{min}}} \right).
\]

Unfortunately the term containing \( \ddot{u}_{rc} \) is bigger than zero and for this reason it cannot be concluded that Equation 4.12 is smaller than zero. However, since \( \ddot{u}_{rc} \) is a lower and upper bounded smooth signal, stability can be proven by using Theorem 1 of [6] in which the nominal system only has to be GUS. This means that the solution stays close to the origin, but will not end up there when time goes to infinity. Taking into account the derivative of the Lyapunov function once more, it can be shown that the solution stays in a one dimensional ball close to the origin since for big values of \( |v_r| \) the second term will dominate the third one, creating a bound where \( \dot{V} \) becomes negative. The bound then is derived from

\[
-\Delta \frac{(\alpha - 1) \alpha}{U_{\text{min}}} |v_r|^2 + \kappa \frac{|V_g|u_{rc}}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r| \leq 0
\]

resulting in a decreasing Lyapunov function for

\[
|v_r| \geq \frac{\kappa u_{rd} |V_g|}{\Delta (\alpha - 1) \alpha \sqrt{U_{\text{min}}^2 - V_y^2}}
\]

\[
\geq \frac{\Delta u_{rd} |V_g|}{U_{\text{min}} (Y_{\text{min}} \Delta - X_{\text{max}} \Gamma_{\text{max}}) \sqrt{U_{\text{min}}^2 - V_y^2}} + \frac{\Delta u_{rd} |V_g|}{(U_{\text{min}} Y_{\text{min}} \Delta - (U_{\text{min}} + U_{\text{max}}) X_{\text{max}} \Gamma_{\text{max}}) \sqrt{U_{\text{min}}^2 - V_y^2}}
\]

(4.15)

For this condition assumption 1 is satisfied and the cascaded system will be stable. This can be concluded because assumption 2 and 3 are automatically satisfied since nothing has changed compared to the case of constant desired surge velocity.

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Chapter 5

Proof stability formation

After converging to all separate paths, the vessels have to move as a single formation, which is defined by desired inter-vessel distances in the along path direction. Reaching these distances is achieved by acceleration and deceleration of the single vessels, which eventually will result in achievement of control objective 2.6 meaning that the vessels move in formation. Moving in formation automatically implies moving in a stable formation and in the sequel of this section a proof is given for stability of path following for the complete system in formation.

The system is defined by sway dynamics \[ \dot{\varepsilon} \varepsilon \varepsilon_r \] T, tracking error dynamics \[ \dot{\xi} \] and along path dynamics \[ \dot{x} \], which are reformulated by a coordinate transformation to \[ \dot{\theta} \]. The underlined values represent a vector with on the \( j \)th entry the value of vessel \( j \), resulting in

\[
\begin{bmatrix}
\dot{\varepsilon} \\
\dot{\varepsilon}_r
\end{bmatrix}
= A(\varepsilon, u_{rc})
\begin{bmatrix}
\varepsilon \\
\varepsilon_r
\end{bmatrix}
+ \begin{bmatrix}
0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & \Omega \left( \frac{2aV \Delta_j}{\pi u_{rcj} \sqrt{(\Delta_j^2 + e_j^2)u_{rcj}^2 - \Delta_j^2V_y^2}} \right)
\end{bmatrix}
\begin{bmatrix}
0_{n \times 1} \\
\Omega \left( \frac{\Delta_i}{\sqrt{(\Delta_i^2 + e_j^2)u_{rcj}^2 - \Delta_i^2V_y^2}} \right)
\end{bmatrix}
+ H'(\varepsilon, v_r, u_{rc}, \xi, \dot{\theta})
\]

\[
\dot{\theta} = -g (L\dot{\theta}) \cos(\psi_{df}) - 2\Omega \left( \sin \left( \frac{\psi_j}{2} \right) \right) \Omega \left( \cos \left( \frac{\psi_j + \psi_{df}}{2} \right) \right) v_r + H_\theta
\]

\[
\dot{\xi} = \Lambda \xi
\]

(5.1)

For the coordinate transformation for the along path dynamics an assumption has to be made that all vessels are influenced by the same ocean current, meaning that they all will end up with the same desired angle \( \psi_{df} \) as will be explained further on. The matrix \( A(\varepsilon, u_{rc}) \) is defined as

\[
A(\varepsilon, u_{rc}) = \begin{bmatrix}
\Omega \left( \frac{\Delta_i}{\sqrt{(\Delta_i^2 + e_j^2)u_{rcj}^2 - \Delta_i^2V_y^2}} \right) & 0_{n \times 1} \\
\Omega \left( \frac{X(u_{rcj})\Gamma_{i}u_{rcj}\Delta_j}{\sqrt{(\Delta_j^2 + e_j^2)u_{rcj}^2 - \Delta_j^2V_y^2}} \right) & \Omega \left( \frac{Y(u_{rcj}) - X(u_{rcj})\Delta_j^2}{\sqrt{(\Delta_j^2 + e_j^2)u_{rcj}^2 - \Delta_j^2V_y^2}} \right)
\end{bmatrix}
\]

(5.2)

and the terms that vanish for \( \xi \to 0 \) are stored in \( H' \)

\[
H' = H_{e,v,\xi} + \Omega \left( \frac{2aV \Delta_1}{\pi u_{rcj} \sqrt{(\Delta_j^2 + e_j^2)u_{rcj}^2 - \Delta_j^2V_y^2}} \right) (I + \Omega(\delta_j \Omega)^2)^{-1} LH_\theta
\]

(5.3)
with

$$H_{e,v_r} = \begin{bmatrix} -X(u_r) & 1 & 0 & 0 \\ \mu_j \Delta_j + c_j & \Gamma_j \Delta_j & 1 & 0 \end{bmatrix} \begin{bmatrix} h_e^T \\ h_v^T \end{bmatrix}$$

(5.4)

$$H_\theta = -2\Omega \left( \sin \left( \frac{\psi_j + \psi_d}{2} \right) \right) \Omega \left( \sin \left( \frac{\tilde{\psi}_j}{2} \right) \right) u_r d - g(L\theta) \left( \cos(\psi_d) \left( \Omega(\cos(\tilde{\psi}_j)) - I \right) - \sin(\psi_d) \Omega(\sin(\tilde{\psi}_j)) \right).$$

(5.5)

For completeness $h_e$ and $h_v$ are given in section 4.1 and for sake of simplicity diagonal matrices are defined by $\Omega(\omega_j)$ with $\omega_j$ the $j^{th}$ element on the diagonal. Furthermore $L_j$ is the $j^{th}$ row of $L$, which is the Laplacian matrix corresponding to the communication digraph $\mathcal{G}$. To meet the stability criterion, the overall system has to satisfy a couple of criteria which are elaborated in detail in the sequel of this chapter.

1. $\xi$ is GES
2. $[\dot{\xi} \ \dot{v_r}]^T$ is bounded regardless $\theta$
3. The cascade with nominal system $\dot{\theta}$ and perturbing system $\dot{\xi}$ is GUAS and ULES
4. The cascade with nominal system $[\dot{\xi} \ \dot{v_r}]^T$ and perturbing systems $\dot{\theta}$ and $\dot{\xi}$ is GUAS.

5.1 GES tracking error dynamics

The tracking error dynamics are defined by

$$\dot{\xi} = \begin{bmatrix} \Omega \left( -(k_u + \frac{d_{11}}{m_{11}}) \right) & 0 & 0 \\ 0 & 0 & I \\ 0 & \Omega(-k_p) & \Omega(-k_r) \end{bmatrix} \xi$$

(5.6)

with a matrix that is Hurwitz. From this it automatically follows that the system is globally exponentially stable.

5.2 Boundedness in sway dynamics

The overall system of sway dynamics and along path dynamics is a feedback interconnected system, since $v_r$ depends on $\theta$ and $\dot{\theta}$ is dependent on $v_r$. This means that both are stable when the other one behaves properly and does not grow unbounded. Since this ends up in a circular reasoning a new strategy is defined. If one of both dynamics, $v_r$ in this case, shows boundedness regardless the behavior of the variable of the other dynamics, $\theta$, the bounded variable $v_r$ will not blow up the other dynamics $\dot{\theta}$ in finite time. To show boundedness a Lyapunov function is defined for $[\dot{\xi} \ \dot{v_r}]^T$ and the terms $\Pi_1$ and $\Pi_2$ are introduced to make the dependency of the interconnection term on $v_r$ more explicit

$$\begin{bmatrix} \dot{\xi} \\ \dot{v_r} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + \Pi_1 \end{bmatrix} \begin{bmatrix} \xi \\ v_r \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ \Pi_2 g(L\theta) \end{bmatrix} + H'(\xi, v_r, u_r, \xi, \theta)$$

(5.7)
The Lyapunov function candidate for this system simply is defined by

$$V = \frac{1}{2}v^2 + \frac{1}{2}r^2$$

which in matrix form looks like

$$V = \frac{1}{2}v^2 + \frac{1}{2}r^2$$

with $$\Pi$$ given in Equation 5.1.

Now the actual bound can be given by a Lyapunov function

$$\dot{V}(t, z, \xi) \leq \|\xi(t)\|$$

which after integration of both sides on the interval $$t_0$$ to $$t$$ results in

$$\frac{1}{2(\Pi_1 + C_1)} \ln \left( \frac{(2\Pi_1 + C_1) V(t, z(t)) + C_2}{(2\Pi_1 + C_1) V(t_0, z(t_0)) + C_2} \right) \leq \int_{t_0}^{t} \|\xi(\tau)\| d\tau.$$  (5.12)

Now the actual bound can be given by a Lyapunov function

$$V(t, z(t)) \leq \left( \frac{(2\Pi_1 + C_1) V(0) + C_2}{(2\Pi_1 + C_1)} \right) e^{(2\Pi_1 + C_1)\lambda} - C_2$$  (5.13)
with \( V(0) = V(t_0, z(t_0)) \) a constant and \( \lambda \) strictly positive. It can be concluded that the right hand side is a positive constant, meaning that \( V(t, z(t)) \) is uniformly bounded for all \( t \geq t_0 \). Since the Lyapunov function is proper and continuous it is concluded that the sway dynamics are globally uniformly bounded for all \( t \geq t_0 \).

### 5.3 Stable along path dynamics in formation

To study the along path dynamics in formation, first of all the circumstances are formulated in which a stable formation is created. To achieve this the vessels need to move with the same surge velocity at the desired beforehand formulated inter-vessel distances. By applying control objective 2.7 and assuming that the ocean current is the same for all vessels, it can also be concluded that all vessels end up with the same desired angle \( \psi_d \) equal to \( \psi_{df} \) and thus the tracking error of the angle can be reformulated to

\[
\tilde{\psi}_j = \psi_j - \psi_{df}.
\] (5.14)

By making use of this new definition a coordinate transformation

\[
\theta_j = x_j - x_d - \int_{t_0}^{t} (u_{rd}(s) \cos(\psi_{df}(s)) - v_{rj} \sin(\psi_{df}(s))) \, ds
\] (5.15)

can be applied in which holds \( x_d - x_i = x_{di} \). This results in the single vessel along path dynamics (which can be interpreted as along path velocity minus desired along path velocity)

\[
\dot{\theta}_j = \dot{x}_j - u_{rd}(\cos(\psi_j) - \cos(\psi_{df})) + \dot{v}_{rj}(\sin(\psi_j) - \sin(\psi_{df}))
\]

\[
= -2u_{rd} \sin\left(\frac{\psi_j + \psi_{df}}{2}\right) \sin\left(\frac{\tilde{\psi}_j}{2}\right) - 2v_{rj} \sin\left(\frac{\tilde{\psi}_j}{2}\right) \cos\left(\cos(\tilde{\psi}_j) - 1\right) - \sin(\psi_{df}) \sin(\tilde{\psi}_j)
\]

\[
= -2u_{rd} \sin\left(\frac{\psi_j + \psi_{df}}{2}\right) \sin\left(\frac{\tilde{\psi}_j}{2}\right) - 2v_{rj} \sin\left(\frac{\tilde{\psi}_j}{2}\right) \cos\left(\cos(\tilde{\psi}_j) - 1\right) - \sin(\psi_{df}) \sin(\tilde{\psi}_j)
\]

with \( h_\theta \) including terms that vanish for \( \tilde{\psi}_j = 0 \)

\[
h_\theta = g \left( \sum_{i \in A_j} (\theta_j - \theta_i) \right) \cos(\psi_{df}) \left( \left( \cos(\tilde{\psi}_j) - 1\right) - \sin(\psi_{df}) \sin(\tilde{\psi}_j) \right).
\] (5.17)

The next step is to prove that the along path dynamics \( \dot{\theta} \) are stable whenever perturbed by the tracking error dynamics \( \dot{\xi} \). To describe this situation a cascade is formulated with \( \dot{\theta} \) the nominal system and \( \dot{\xi} \) the perturbing system. Now Theorem 2 of [6], defined in section 4.1, can be applied to the system

\[
\begin{align*}
\dot{\theta} &= -g \left( \mathcal{L}_\theta \right) \cos(\psi_{df}) + H_\theta \\
\dot{\xi} &= \Lambda \xi
\end{align*}
\] (5.18)

with \(-2\Omega \left( \sin\left(\frac{\tilde{\psi}_j}{2}\right) \right) \Omega \left( \cos\left(\frac{\psi_j + \psi_{df}}{2}\right) \right) \nu_r \) included in \( H_\theta \). This is feasible because \( \tilde{\psi}_j \) will drive the term to zero when it goes to zero itself, because of the boundedness of all other
variables in the term. The equilibrium point of along path dynamics is reached when all \(\theta_j\) values for the \(n\) vessels converge to the same value, making \(\dot{\theta} = 0\). This however is not the right format for Theorem 2 of [6], since the system has to converge to the origin to prove asymptotic stability. To make 5.18 suited for the proof, Lemma 2 of [1] is used to conclude the existence of a coordinate transformation to the right format, formulated here as Lemma 1.

**Lemma 1:** Consider system 5.18, there exists a coordinate transformation \(\phi = T\theta, \ T \in \mathbb{R}^{(n-1) \times n}\), such that the following holds:

1. \(\phi = 0\) implies that \(\theta_1 = \ldots = \theta_n\).
2. The format of the along path dynamics looks like

\[
\dot{\phi} = f(\phi) + G(\xi, u_{rc})\xi
\]

with \(G(\xi, u_{rc})\) a globally bounded function, uniformly in \(\xi\) and \(u_{rc}\).

3. The system \(\dot{\phi} = f(\phi)\) is GUAS and ULES with positive definite and radially unbounded Luapunov function \(V(\phi)\) satisfying

\[
\frac{\partial V(\phi)}{\partial \phi} f(\phi) \leq -W(\phi) < 0, \quad \forall \phi \in \mathbb{R}^{(n-1)}
\]

(5.20)

Applying this coordinate transformation to the along path dynamics, Lemma 3 of [1] can be used to conclude the stability properties of system \(\phi\) perturbed by \(\dot{\xi}\), here formulated as Lemma 2.

**Lemma 2:** Considering the system as described in previous sections, the cascaded system 5.18 is GUAS and ULES. Taking the time derivative of the positive definite and radially unbounded function \(V(\phi)\) from lemma 1.3 along the solutions of \(\phi\) and combining this with Equation 5.20 results in

\[
\dot{V} = \frac{\partial V(\phi)}{\partial \phi} f(\phi) + \frac{\partial V(\phi)}{\partial \phi} G(\xi, u_{rc})\xi
\]

\[
\leq -W(\phi) + C_\theta C_\xi \|\xi(t)\|
\]

(5.21)

In the second step use is made of the definition \(\|G(\xi, u_{rc})\| \leq C_\xi\) with \(C_\xi > 0\). This is feasible because \(G(\xi, u_{rc})\) is uniformly bounded, as also defined in lemma 1.2. Since \(\xi(t)\) satisfies the integrability condition

\[
\int_{t_0}^{\infty} \|\xi(s)\| \, ds \leq \gamma (\|\xi(t_0)\|)
\]

(5.22)

as was worked out for the single vessel in section 4.1, this can be combined with the integration of Equation 5.21, which results in

\[
V(\phi(t)) \leq V(\phi(t_0)) + C_\theta C_\xi \gamma (\|\xi(t_0)\|), \quad t \geq t_0.
\]

(5.23)
From this it is directly concluded that $V(\phi)$ is uniformly bounded. Furthermore, since $V$ itself is radially unbounded it follows that $\phi(t)$ is uniformly bounded. Since it also holds that the nominal system $\dot{\phi} = f(\phi)$ is GUAS and ULES and the solutions of 5.19 are uniformly bounded under perturbation of UGES system $\dot{\xi}$, it follows from [7] and [8] that the cascade of 5.18 is GUAS and ULES.

Finally it is concluded that lemma2 defines the origin $(\phi, \xi) = (0, 0)$ to be GUAS and ULES, which, after applying lemma1, means that $\theta_j, j = 1, ..., n$, will converge to the same value exponentially fast for any ball of initial conditions.

### 5.4 GUAS overall system

Since it is proven now that the along path dynamics behave asymptotically stable under perturbation of the tracking error dynamic and that the sway dynamics behave bounded under perturbation of the along path dynamics, asymptotic stability of the overall system has to be proven. For this a cascade is formulated with the sway dynamics $[\dot{\xi} \, \dot{\psi}]^T$ as nominal system and the tracking error dynamics $\dot{\xi}$ and along path dynamics $\dot{\theta}$ both as perturbing systems, which is given by

$$
\begin{bmatrix}
\dot{\xi} \\
\dot{\psi}
\end{bmatrix} = A(\xi, u_{r_s}) \begin{bmatrix}
\xi \\
\psi
\end{bmatrix} + H'(\xi, \psi, u_{r_s}, \xi, \theta) \\
\dot{\theta} = -g(L\theta) \cos(\psi_d) + H_\theta \\
\dot{\xi} = \Lambda \xi.
$$

(5.24)

The terms $\Pi_1 \|v_r\|^2$ and $\Pi_2 g(L\theta)$ are placed in $H'(\xi, \psi, u_{r_d}, \xi, \theta)$ since they vanish for $\xi$ and $\dot{\theta}$ going to zero, which is the case since both perturbing systems are GUAS and ULES. To prove stability of the cascade, theory 2 of [6] is used as explained before in section 4.1 and to meet assumption 1 the same Lyapunov function as defined in that section can be used, since the structure of the nominal system is exactly the same. Next to this assumption 2 is met since the vanishing terms at most are linear in $\xi$ and $\psi$, which limits the growth. The integrability condition also is met for both perturbing systems, since $\dot{\theta}$ is ULES and $\dot{\xi}$ is UGES, which is the condition for assumption 3. By satisfying all three conditions it is concluded that the overall system is GUAS.
Chapter 6
Simulations

The model has been validated by simulations in MATLAB, which take into account the path following for the single vessel and the formation control for multiple vessels which can be extended to a desired amount of participants. For ease of the simulation the vessels are assumed to be all identical, however in practice there is no need for this. The dynamics of the vessel with a mass of \( m = 6.4 \cdot 10^6 \text{ kg} \) and length \( L = 76.2 \text{ m} \) are defined by

\[
M = \begin{bmatrix}
7.22 \cdot 10^6 & 0 & 0 \\
0 & 1.21 \cdot 10^7 & -3.63 \cdot 10^7 \\
0 & -3.63 \cdot 10^7 & 4.75 \cdot 10^9
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & -1.21 \cdot 10^7 v_r + 3.63 \cdot 10^7 \\
1.21 \cdot 10^7 v_r - 3.63 \cdot 10^7 r & 7.22 \cdot 10^6 u_r & 0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
95070 & 0 & 0 \\
0 & 4.34 \cdot 10^6 & -2.27 \cdot 10^6 \\
0 & -1.88 \cdot 10^7 & 7.57 \cdot 10^8
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1.13 \cdot 10^6 & 0 \\
0 & 9.63 \cdot 10^7
\end{bmatrix}
\]

Furthermore some settings are given to the design parameter \( \Delta \) (look ahead distance) which is of arbitrary choice when it satisfies its lower bound as given in Equation 4.13. Also the external inputs from the ocean current, \( V_x \) and \( V_y \), are defined on a constant value during the simulations.

- \( \Delta = 300 \text{ m} \)
- \( V_x = 1 \text{ m/s} \)
- \( V_y = 3 \text{ m/s} \)

Next to this all the control gains that are formulated to be positive are set equal to one and \( \dot{\psi}_d \) and \( \dot{u}_r \) are set to zero since the definitions are not formulated explicitly. This only affects the controllers in Equation 3.10, meaning that the actual accelerations are not taken into account while controlling. Although the observer is not included in the proof worked out in Chapter 5, it is used in the simulations to check its influence. Including the observer makes the control technique better useful and widely applicable, making the inclusion sufficient valuable.
6.1 Single vessel

For a single vessel the model is evaluated in an ideal situation, meaning that the vessel starts from standstill and that the ocean current is estimated exactly correct. This practically means that the observer provides the exact values for the ocean current during the complete simulation and for this reason the proof of Chapter 5 is still applicable. Furthermore the (constant) desired surge velocity for the single vessel is given by $u_{rd} = 5\text{m/s}$ and its initial position is given by $[x_{in} \ y_{in} \ \psi_{in}] = [0 \ 800 \ -\frac{2}{3}\pi]$, resulting in the behavior of Figure 6.1.

$$\text{Figure 6.1: The vessel's behavior during the first 1000 seconds of the path following with all initial conditions ideal, meaning that the vessel starts from standstill in surge and sway direction and that the ocean current is estimated exactly correct.}$$

From this it can be concluded that the controllable surge velocity and yaw angle converge to the desired values exponentially fast. Furthermore the cross track error goes smoothly to zero and the sway dynamics given by the yaw rate and sway velocity converge to zero as well, meaning that the system is asymptotically stable. Next to this the observer values represent the exact values during the complete simulation, as was expected.

Showing the results in a more natural way, the path followed by the vessel is represented in Figure 6.2. Here it can be seen that the vessel moves over a smooth curve, that in fact is the same as the cross track error represented in Figure 6.1. In addition on this the vessel is orientated under the actual angle as represented by the blue solid line in the middle top subplot of Figure 6.1. Although the scale in x- and y- axis is not 1:1, it can be seen that the center point of the bow of the vessel is orientated correctly with respect to the line it moves on. It logically is expected that the vessel points away from the path it is moving on in the...
direction the lateral ocean current is coming from. This is the case and for this reason it is concluded that scaling does not affect the real representation. It however can be said that in the real case the vessel is orientated a bit more away from the line it moves on. Apart from the smooth movement, one comment has to be made. By focusing on the first seconds of the simulation, as represented in Figure 6.3, it can be seen that the vessel experiences some settling issues during the start-up in which the vessel moves backwards.

The fact that the vessel is driven away during the first seconds can be explained by the initial surge velocity of $0\text{ m/s}$. While the vessel starts moving in the opposite direction of the ocean current with a non-desired angle, the ocean current is already moving it away from its initial position. This happens because the velocity of the ocean current is bigger than the absolute value of the surge velocity. Furthermore it can be concluded that the vessel is directed to the track very smoothly and that the settling issues do not cause problems when it is taken into account that the first meters can be traveled in an undesired direction. Additionally the undesired path is only $1.5\text{ m}$, which most likely is negligible on an initial error of $800\text{ m}$.

### 6.2 Multiple vessels

By taking into account the relations between the vessels as mentioned before, the simulation for one vessel can be extended to a simulation for multiple vessels that have to operate in a certain formation. In this report a formation of five vessels is worked out with the individual tracks located at $[D_1 \ D_2 \ D_3 \ D_4 \ D_5] = [-1000 \ 0 \ 250 \ 800 \ 1500]$ and inter-vessel distances $[d_{12}\ d_{23}\ d_{34}\ d_{45}] = [0 \ 0 \ 0 \ 0]$. This describes a situation of vessels moving on a non-equal side-away distance moving forward as a straight line.
**Figure 6.4:** The center vessels behavior during the first 1500 seconds of the path following with all initial conditions ideal, meaning that the vessel starts from standstill and that the ocean current is estimated exactly correct.

**Figure 6.5:** Vessels starting in all different conditions are made to follow a line and align to each other to make a straight-line formation. Forming this takes 4500 m.
Figure 6.4 represents the behavior of the center (third) vessel with in the center bottom subplot the inter-vessel distances with respect to this vessel. In Figure 6.5 it can be seen that all vessels start at the same along path position \( x = 0 \), however due to different orientations and differences in the initial cross track error, it easily can be seen that vessel 1 will get behind and vessel 5 will take the lead. This is confirmed by the inter-vessel distances. To compensate for this the commanded surge velocity for vessel 1 will be higher than the desired value and the one for vessel 5 will be lower. In the top right subplot of Figure 6.4 it can be seen that also the commanded surge velocity of vessel 3 is adapted to form the formation, which in 4500\( m \) is achieved regarding Figure 6.5. The two times \( ur_c \) switches sign results in small discontinuities in the sway dynamics and in the yaw angle. This however is bounded, meaning that the vessel behaves properly regardless accelerations in \( ur_c \) as was proven in chapter 5.

Besides aligning vessels in a straight line formation all possible formations can be created since the wide area in the ocean gives all opportunities for that. A convenient operating structure is defined in Figure 6.6 in which all vessels move on an equal side-away distance with an inverse v-shape. Since more acceleration and deceleration is needed to achieve this structure, the formation is achieved after 13000\( m \). This is significantly more than in the previous case and when needed the settling distance can be made smaller by increasing the maximum difference in commanded surge velocities for vessels that have to slow down and speed up. This now is defined on 5% of \( ur_d \), being 0.25\( m/s \).

**Figure 6.6:** Vessels starting in all different conditions are made to follow their prescribed tracks which are equally distributed over the surface. This time the formation is an inverse V and it takes 13000\( m \) to align to the formation.
6.3 Start-error observer value ocean current

By defining the initial values of the observer as $\hat{V}_x = 10$ and $\hat{V}_y = -3$, its influence on the vessels behavior is checked as well as the robustness of the system. Taking into account the vessels behavior in Figure E.1 in Appendix E, it can be concluded that a small deviation from the real value of the ocean current is canceled out by the controller. The only undesirable thing is that the vessel will move in a non-optimal direction in the first meters as is shown in Figure 6.8, however afterwards it will converge to the track it has to follow as is represented by Figure 6.7. Remarkably the undesired direction is better than the desired direction here, however this is not the general case. Checking all other parameters in Figure E.1 it is concluded that the system is asymptotically stable and that the introduction of the observer does not affect the stability of the controlled system. Although in this report no formal proof is formulated including the observer, the system is supposed to be stable regarding the simulation results.

![Figure 6.7: Despite the difference between the ocean current and its estimate the vessel moves smoothly to the track it has to follow.](image1)

![Figure 6.8: By zooming in it can be seen that for the first seconds the vessel is correcting for the wrong estimated ocean current. Remarkably for this case it means that the vessel starts moving in a better direction than in the case with an exact estimated ocean current.](image2)
Chapter 7

Conclusion

A control technique is designed based on the Line Of Sight guidance law to let a single vessel follow a straight line path with the disturbance of an estimated ocean current. For this a Luenberger observer is used and the controller first is designed for a vessel that follows the path with a constant desired surge velocity. By assuming that the ocean current is determined exactly by measurements instead of an observer, the controller is proven to be GUAS in a straight forward way. Including the observer will make the proof more complicated and for this reason is omitted.

To make the path following control technique applicable for multiple vessels in formation, the technique for the single vessel is extended. The constant desired surge velocity is enabled to be variable, since this is needed to correct for errors in the formation structure. Proving stability for this results in ultimate boundedness with a bound bigger than zero, meaning that the vessel will move around the desired track and thus is not GUAS.

Applying this to multiple vessels a synchronization controller is designed that makes the desired surge velocity dependent on the inter-vessel distances. Using this in the overall dynamics of the system the vessels can be formulated as cascaded systems with the sway dynamics as the nominal system and the tracking error dynamics as an perturbing system as well as the along path dynamics as a perturbing system. This is proven to be GUAS and ULES as is also confirmed by the simulations, meaning that the formation is asymptotically stable and that the vessels follow their individual path asymptotically stable. For this the assumption is made that all vessels will end up with the same orientation due to disturbance by the same ocean current.

Furthermore the observer, that was not included in the proof, gives GUAS results in the simulations and from this it is expected that the observer can be included in the proof given in this report. The complete proof however is something to be done in future work and next to this some future work can be done on weakening the assumption that all vessels will end up with the same orientation.
Appendix A

Vessel model

\[
F_{ur}(v_r, r) = \frac{1}{m_{11}} (m_{22} v_r + m_{23} r) r
\]
\[
F_{r}(u_r, v_r, r) = \frac{m_{23} d_{22} - m_{22} (d_{32} + (m_{22} - m_{11}) u_r)}{m_{22} m_{33} - m_{23}^2} v_r
+ \frac{m_{23} (d_{22}) - m_{22} (d_{32} + (m_{22} - m_{11}) u_r)}{m_{22} m_{33} - m_{23}^2} r
\]
\[
X(u_r) = \frac{m_{23}^2 - m_{11} m_{33}}{m_{22} m_{33} - m_{23}^2} u_r + \frac{d_{33} m_{23} - d_{23} m_{33}}{m_{22} m_{33} - m_{23}^2}
\]
\[
Y(u_r) = \frac{(m_{22} - m_{11}) m_{23}}{m_{22} m_{33} - m_{23}^2} u_r - \frac{d_{22} m_{22} - d_{32} m_{23}}{m_{22} m_{33} - m_{23}^2}
\]

Figure A.1: A communication network with the dots representing vessels and the arrows representing information transport. A is a globally reachable vertex since its information is received by all other vessels.
Appendix B

Controller design

\[
h_x = \begin{bmatrix}
\frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} \cos \tilde{\psi} + \frac{\epsilon + \mu \Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} \sin \tilde{\psi} \\
- \frac{u_r \Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} - \frac{v_r (\epsilon + \mu \Delta)}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} \sin \tilde{\psi} & \sin \tilde{\psi} & \frac{u_r (\epsilon + \mu \Delta)}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \frac{v_r \Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & 0
\end{bmatrix}
\]

(B.1)
Appendix C

Stability proof for path following for a single vessel

C.1 Constant desired surge velocity $u_{rd}$

This section contains the derivation of the derivative of the Lyapunov function $V = \frac{1}{2} e^2 + \frac{1}{2} \kappa v_r^2$, which is given by

$$
\dot{V} = e \dot{e} + \kappa v_r \dot{v}_r
$$

$$
= -\frac{u_{rd} e^2}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \frac{\Delta e v_r}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \kappa \frac{X(u_{rd}) \Gamma u_{rd} \Delta e v_r}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \kappa \left(Y(u_{rd}) - \frac{X(u_{rd}) \Gamma^2}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}\right) v_r^2.
$$

(C.1)

This can be simplified by making use of two reformulations

$$
Y < 0 \rightarrow Y \leq -Y_{\min}
$$

(C.2)

$$
(\pm) X \leq |X| \leq X_{\max}
$$

(C.3)

in which $X(u_{rd})$ and $Y(u_{rd})$ are simplified to $X$ and $Y$ in the notation. The simplifications do not affect the physical meaning of the equation, as also can be said for using absolute values for $|e|$ and $|v_r|$ since the problem is symmetrically. Implementing this gives

$$
\dot{V} \leq -\frac{u_{rd} e^2}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \frac{\Delta e |v_r|}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} + \kappa \frac{X_{\max} \Gamma u_{rd} \Delta |e||v_r|}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}
$$

$$
+ \kappa \left(-Y_{\min} + \frac{X_{\max} \Gamma \Delta^2}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}\right) v_r^2.
$$

(C.4)

After introducing $z = \frac{|e|}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}$ and defining two more simplifications

$$
-\sqrt{(\mu \Delta + e)^2 + \Delta^2} \leq -\Delta
$$

(C.5)

$$
\frac{1}{(\mu \Delta + e)^2 + \Delta^2} \leq \frac{1}{\Delta^2}
$$

(C.6)
\( \dot{V} \) can be formulated as
\[
\dot{V} \leq -u_{rd} \Delta z^2 + \left( \Delta + \kappa \frac{X_{\text{max}} \Gamma u_{rd}}{\Delta} \right) z |v_r| + \kappa \left( -Y_{\text{min}} + \frac{X_{\text{max}} \Gamma}{\Delta} \right) v_r^2. \tag{C.7}
\]

With the aim for further simplification, the variable \( \alpha \) is introduced to simplify term 2, making \( \kappa \) a function of \( \alpha \)
\[
2\alpha \Delta = \Delta + \kappa \frac{X_{\text{max}} \Gamma u_{rd}}{\Delta} \\
\kappa = \frac{(2\alpha - 1) \Delta^2}{X_{\text{max}} \Gamma u_{rd}}. \tag{C.8}
\]
which only leaves the third term to be reduced further
\[
\dot{V} \leq -u_{rd} \Delta z^2 + 2\alpha \Delta z |v_r| + \kappa \left( -Y_{\text{min}} + \frac{X_{\text{max}} \Gamma}{\Delta} \right) v_r^2 \tag{C.9}
\]
such that the Lyapunov function gets the form
\[
\dot{V} \leq -\Delta \left( \sqrt{u_{rd} z} - \frac{\alpha |v_r|}{\sqrt{u_{rd}}} \right)^2 - \Delta \frac{(\alpha - 1)\alpha}{u_{rd}} |v_r|^2 \tag{C.10}
\]
which clearly is smaller than zero. Achieving this is done by defining \( \alpha \) as
\[
\alpha = \frac{Y_{\text{min}} \Delta - X_{\text{max}} \Gamma}{X_{\text{max}} \Gamma}
\]
from which the lower bound can be extracted for the look-ahead-distance
\[
\Delta > 2 \frac{X_{\text{max}} \Gamma}{Y_{\text{min}}}
\]
with the extra condition \( \alpha > 1 \). Unfortunately \( \Gamma \) is a function of \( \Delta \) which not simply can be isolated from all terms. To deal with this \( \Gamma \) is reformulated for an extreme case in which the error goes to infinity, leading to a saturated maximum value and a new definition for the look-ahead distance
\[
\Gamma_{\text{max}} = \frac{u_{rd} (V_y + u_{rd})}{u_{rd}^2 - V_y^2} \\
\Delta > 2 \frac{X_{\text{max}} \Gamma_{\text{max}}}{Y_{\text{min}}}. \tag{C.11}
\]

C.2 Variable desired surge velocity \( u_{rc} \)

This section contains the derivation of the derivative of the Lyapunov function \( V = \frac{1}{2} e^2 + \frac{1}{2} \kappa v_r^2 \), which is similar to Equation C.1 with an extra term containing \( \dot{u}_{rc} \)
\[
\dot{V} \leq -\frac{|V_y| e^2}{(\mu \Delta + \epsilon)^2 + \Delta^2} + \frac{\Delta |e| |v_r|}{(\mu \Delta + \epsilon)^2 + \Delta^2} + \kappa \frac{X_{\text{max}} \Gamma U_{\text{max}} \Delta |e| |v_r|}{(\mu \Delta + \epsilon)^2 + \Delta^2} \\
+ \kappa \left( -Y_{\text{min}} + \frac{X_{\text{max}} \Gamma \Delta^2}{(\mu \Delta + \epsilon)^2 + \Delta^2} \right) v_r^2 + \kappa \frac{|V_y| u_{rd}}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r|. \tag{C.12}
\]
Introducing $z$ and taking into account the simplifications of C.5 and C.6 enables to reformulate $\dot{V}$ to

$$
\dot{V} \leq -|V_y|\Delta z^2 + \left( \Delta + \frac{X_{\text{max}} \Gamma_{\text{max}} U_{\text{max}}}{\Delta} \right) \kappa (\alpha U_{\text{min}} + \frac{X_{\text{max}} \Gamma}{\Delta}) v_r^2 + \Delta \kappa \frac{V_y |\bar{u}_{rd}|}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r|.
$$

(C.13)

Defining $\kappa$ as C.8, with in this case $u_{rd} = U_{\text{max}}$, leads to

$$
\alpha = \frac{U_{\text{min}} (Y_{\text{min}} \Delta - X_{\text{max}} \Gamma)}{U_{\text{max}} X_{\text{max}} \Gamma}
$$

to achieve the Lyapunov function of C.10 with the last term of C.13 not taken into account. From this a new lower bound can be defined for the look-ahead-distance

$$
\Delta > \frac{X_{\text{max}} \Gamma_{\text{max}}}{Y_{\text{min}}} \left( 1 + \frac{U_{\text{max}}}{U_{\text{min}}} \right)
$$

by taking into account $\Gamma_{\text{max}}$. However the term containing $\bar{u}_{rd}$ is bigger than zero and for this reason cannot be neglected. Since $\bar{u}_{rd}$ is a lower and upper bounded smooth signal, stability can be proved by using Theorem 1 of [6] in which the nominal system only has to be GUS. This means that the solution stays close to the origin, but will not end up there when time goes to infinity. By formulating the Lyapunov function in the format of C.10

$$
\dot{V} \leq -\Delta \left( \sqrt{U_{\text{min}}^2} - \frac{\alpha |v_r|}{\sqrt{U_{\text{min}}}} \right)^2 - \Delta (\alpha U_{\text{min}} + \frac{X_{\text{min}} \Delta}{\Delta}) |v_r|^2 + \Delta \kappa \frac{|V_y| |\bar{u}_{rd}|}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r|
$$

(C.14)

it can be shown that the solution stays in a one dimensional ball close to the origin since for big values of $|v_r|$ the second term will dominate the third one. The bound than is defined by

$$
-\Delta (\alpha - 1) U_{\text{min}} |v_r|^2 + \Delta \kappa \frac{|V_y| |\bar{u}_{rd}|}{U_{\text{min}} \sqrt{U_{\text{min}}^2 - V_y^2}} |v_r| \leq 0
$$

(C.15)

resulting in a decreasing Lyapunov function for

$$
|v_r| \geq \frac{\Delta |\bar{u}_{rd}| |V_y|}{(\alpha - 1) \alpha \sqrt{U_{\text{min}}^2 - V_y^2}}
$$

$$
\geq \frac{\left( 2 U_{\text{min}} \left( \frac{Y_{\text{min}} \Delta}{X_{\text{max}} \Gamma_{\text{max}}} - 1 \right) - U_{\text{max}} \right) \Delta U_{\text{min}} |\bar{u}_{rd}| |V_y|}{U_{\text{min}} \left( \frac{Y_{\text{min}} \Delta}{X_{\text{max}} \Gamma_{\text{max}}} - 1 \right) U_{\text{min}} \left( Y_{\text{min}} \Delta - X_{\text{max}} \Gamma_{\text{max}} \right) \sqrt{U_{\text{min}}^2 - V_y^2}}
$$

$$
\geq \frac{\left( 1 + \frac{U_{\text{min}} \left( \frac{Y_{\text{min}} \Delta}{X_{\text{max}} \Gamma_{\text{max}}} - 1 \right)}{U_{\text{min}} \left( \frac{Y_{\text{min}} \Delta}{X_{\text{max}} \Gamma_{\text{max}}} - 1 \right)} - U_{\text{max}} \right) \Delta U_{\text{min}} |\bar{u}_{rd}| |V_y|}{U_{\text{min}} \left( \frac{Y_{\text{min}} \Delta}{X_{\text{max}} \Gamma_{\text{max}}} - 1 \right) U_{\text{min}} \left( Y_{\text{min}} \Delta - X_{\text{max}} \Gamma_{\text{max}} \right) \sqrt{U_{\text{min}}^2 - V_y^2}}
$$

(C.16)

$$
\geq \frac{\Delta |\bar{u}_{rd}| |V_y|}{U_{\text{min}} \left( Y_{\text{min}} \Delta - X_{\text{max}} \Gamma_{\text{max}} \right) \sqrt{U_{\text{min}}^2 - V_y^2}} + \frac{\Delta |\bar{u}_{rd}| |V_y|}{(U_{\text{min}} Y_{\text{min}} \Delta - (U_{\text{min}} + U_{\text{max}}) X_{\text{max}} \Gamma_{\text{max}}) \sqrt{U_{\text{min}}^2 - V_y^2}}
$$
Appendix D

Proof stability formation

\[ H(e, v_r, u_r, \xi) = \begin{bmatrix} -X(u_r) & \frac{\Gamma \Delta}{(\mu \Delta + e)^2 + \Delta^2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_1(\xi) + H_2(\xi) u_r \end{bmatrix} \] (D.1)

\[ H_1 = \begin{bmatrix} \frac{\frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} \cos \psi + \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} \sin \psi}{\frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}} & \frac{\frac{X(u_r + \bar{u}_r) - X(u_r)}{u_r}}{\frac{\sqrt{(\mu \Delta + e)^2 + \Delta^2}}{u_r}} & 0 \\ \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & \frac{\frac{\Gamma \Delta u_r e}{(\mu \Delta + e)^2 + \Delta^2}}{\frac{\Gamma \Delta u_r e}{(\mu \Delta + e)^2 + \Delta^2}} & 0 \\ 0 & 0 \end{bmatrix} \] (D.2)

\[ H_2 = \begin{bmatrix} \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & 0 \\ \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & \frac{\Delta}{\sqrt{(\mu \Delta + e)^2 + \Delta^2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (D.3)
Appendix E

Simulations

Figure E.1: The vessels behavior during the first 1000 seconds of the path following with a different value for the actual ocean current and the estimates in both directions. It can be seen that this deviation is canceled out by the system. In this case it even turns out to better than correcting for the real value.
Bibliography


